

Cosmic No Hair for Braneworlds with a Bulk Dilaton Field

Article (Published Version)

Lidsey, James E and Seery, David (2005) Cosmic No Hair for Braneworlds with a Bulk Dilaton Field. *Physical Review D*, 72 (10). p. 103507. ISSN 1550-7998

This version is available from Sussex Research Online: <http://sro.sussex.ac.uk/id/eprint/25281/>

This document is made available in accordance with publisher policies and may differ from the published version or from the version of record. If you wish to cite this item you are advised to consult the publisher's version. Please see the URL above for details on accessing the published version.

Copyright and reuse:

Sussex Research Online is a digital repository of the research output of the University.

Copyright and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable, the material made available in SRO has been checked for eligibility before being made available.

Copies of full text items generally can be reproduced, displayed or performed and given to third parties in any format or medium for personal research or study, educational, or not-for-profit purposes without prior permission or charge, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Cosmic no hair for braneworlds with a bulk dilaton field

James E. Lidsey and David Seery

Astronomy Unit, School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London, E1 4NS, UK
(Received 23 September 2005; published 8 November 2005)

Braneworld cosmology supported by a bulk scalar field with an exponential potential is developed. A general class of separable backgrounds for both single and two-brane systems is derived, where the bulk metric components are given by products of world volume and bulk coordinates and the world-volumes represent any anisotropic and inhomogeneous solution to an effective four-dimensional Brans-Dicke theory of gravity. We deduce a cosmic no hair theorem for all ever-expanding, spatially homogeneous Bianchi world volumes and find that the spatially flat and isotropic inflationary scaling solution represents a late-time attractor when the bulk potential is sufficiently flat. The dependence of this result on the separable nature of the bulk metric is investigated by applying the techniques of Hamilton-Jacobi theory to five-dimensional Einstein gravity. We employ the spatial gradient expansion method to determine the asymptotic form of the bulk metric up to third-order in spatial gradients. It is found that the condition for the separable form of the metric to represent the attractor of the system is precisely the same as that for the four-dimensional world-volume to isotropize. We also derive the fourth-order contribution to the Hamilton-Jacobi generating functional. Finally, we conclude by placing our results within the context of the holographic approach to braneworld cosmology.

DOI: [10.1103/PhysRevD.72.103507](https://doi.org/10.1103/PhysRevD.72.103507)

PACS numbers: 98.80.Cq, 04.25.-g, 11.25.-w

I. INTRODUCTION

One of the most striking features of our observable universe is that on sufficiently large scales it is very nearly spatially isotropic and homogeneous. The inflationary paradigm provides an attractive, dynamical mechanism for the universe to evolve into such a symmetric state from a potentially wide class of anisotropic and inhomogeneous initial conditions [1–3]. Such a feature is generally referred to as “cosmic no hair.” Braneworld cosmology, motivated by string/M-theoretic considerations [4], has received considerable attention in recent years. (For reviews, see, e.g., Refs. [5]). In this scenario our four-dimensional universe is viewed as a codimension one brane embedded in a higher-dimensional “bulk” space. It is important, therefore, to investigate the isotropization of the universe within the context of braneworld inflation [6,7].

To date, however, progress in this direction has been hindered by our lack of knowledge of the geometry of the bulk space. In the Randall-Sundrum (RS) scenario [8], for example, our braneworld is embedded in a five-dimensional Einstein space sourced by a negative cosmological constant. In particular, a spatially isotropic braneworld propagates in five-dimensional Anti-de Sitter (AdS)-Schwarzschild space. On the other hand, due to the complexity of the field equations, very few exact anisotropic (or inhomogeneous) solutions to the five-dimensional bulk Einstein equations have been found [7].

A natural and well-motivated extension of the RS scenario is to include one or more scalar fields in the bulk action. In this paper we consider a five-dimensional action for gravity coupled to two scalar fields $\{\varphi, \sigma\}$:

$$S_5 = \int_{\mathcal{M}_5} d^5x \sqrt{-\hat{g}} \left[\hat{R} - \frac{1}{2}(\hat{\nabla}\varphi)^2 - \frac{1}{2}e^{-b\varphi}(\hat{\nabla}\sigma)^2 - V(\varphi) \right] + \sum_{i=1}^2 \int_{\mathcal{M}_4^{(i)}} d^4x \sqrt{-g_i} T_i(\varphi), \quad (1)$$

where the “dilaton” field φ self-interacts through a potential $V(\varphi)$ and is coupled to the branes through its brane potentials $T_i(\varphi)$ that are localized to the branes¹. In the case where the bulk dimension has the topology S^1/Z_2 , we may consider two “end-of-the-world” branes located at the orbifold fixed points. The metrics induced on these four-dimensional hypersurfaces are defined by $g_{\mu\nu}^{(1)} \equiv \hat{g}_{\mu\nu}(y=0)$ and $g_{\mu\nu}^{(2)} \equiv \hat{g}_{\mu\nu}(y=\pi)$, respectively, where the fifth dimension is parametrized by the coordinate y . If the bulk dimension is not periodic, we may consider the case of a single brane by specifying $T_1 \neq 0$ and $T_2 = 0$ such that the Z_2 symmetry is still respected across the brane. The constant b determines the coupling between the dilaton and massless “axion” field, σ .

We will focus on the class of exponential self-interaction potentials:

$$V(\varphi) = V_0 e^{-q\varphi}, \quad T_i(\varphi) = \mu_i e^{-q\varphi/2}, \quad (2)$$

where $\{q, V_0, \mu_i\}$ are constants. An action of the form (1) and (2) is well motivated from a number of perspectives. When $q = 2$, $b = 1$, and $\mu_1 = -\mu_2 = \sqrt{6V_0}$, it repre-

¹The bulk spacetime metric has signature $(-, +, +, +, +)$. Upper case, Latin indices vary from $(0, \dots, 4)$ and lower case, Greek indices span $(\mu = 0, 1, 2, 3)$. A circumflex accent denotes geometric quantities compatible with the five-dimensional metric \hat{g}_{AB} . The Ricci curvature scalar of the bulk spacetime \mathcal{M}_5 is denoted by \hat{R} , $\hat{g} \equiv \det \hat{g}_{AB}$ and $g^{(i)} \equiv \det g_{\mu\nu}^{(i)}$.

sents a consistent truncation of Hořava-Witten theory [9] compactified on a Calabi-Yau threefold, where the dilaton represents the breathing mode of the Calabi-Yau space and the axion arises from the universal hypermultiplet [10]. The potential is generated by the nontrivial flux of the four-form field strength on four-cycles of the Calabi-Yau space. In the absence of an axion field, action (1) and (2) also follows from the toroidal, Kaluza-Klein compactification of the $(5 + m)$ -dimensional RS model, where the dilaton again represents the breathing mode of the internal dimensions and the exponential coupling is given by $q = \sqrt{2m/[3(m+3)]}$ [11].

Recently, cosmological solutions with a single brane [12] and a two-brane configuration [13] were found for action (1) and (2) with a vanishing axion field when the world volume of the branes is represented by the spatially flat and isotropic Friedmann-Robertson-Walker (FRW) metric. In both cases, the five-dimensional bulk solution has the form

$$ds^2 = n^2(y)[-dt^2 + t^{4/(3q^2)}\delta_{ij}dx^i dx^j + t^2 dy^2], \quad (3)$$

where $n \propto e^{2\psi/3q^2}$, $e^{-q\varphi} \propto t^{-2}e^{-2\psi}$, and $\psi = \psi(y)$ is a function of y that is determined by solving the field equations. This solution may be interpreted by an observer confined to the brane as a power-law inflationary cosmology for $q^2 < 2/3$ and inflation proceeds as the interbrane distance increases. Solution (3) is “separable,” in the sense that physical quantities are represented as products of functions of t and y . This corresponds physically to the case where there is no net propagation of scalar waves in the bulk.

The solution (3) also represents a scaling solution, since the Hubble parameter on the brane and the dilaton’s kinetic energy scale at the same rate. In general, scaling solutions play an important role in cosmology. They establish the asymptotic behavior of a particular cosmology as well as determining its stability properties. Moreover, the attractive nature of scaling solutions provides a dynamical framework where the initial conditions for any subsequent cosmological evolution can be well defined. Mukohyama and Coley [13] have shown that in the two-brane scenario the scaling solution (3) is stable against homogeneous linear metric perturbations and have also found that it represents the late-time attractor for a general FRW world volume. Recently, however, an unstable mode has been identified in the case where the position of one of the branes is perturbed without producing a corresponding metric perturbation and this can result in a brane collision [14].

The purpose of the present work is to investigate the implications of relaxing the separable ansatz discussed above as well as the assumption that the world-volume corresponds to an isotropic FRW metric. In Sec. II, we first consider the case of an arbitrary world-volume metric and show that, in general, the effective dynamics on the brane

can be described by a Brans-Dicke scalar-tensor theory of gravity, where the coupling between the four-dimensional dilaton and graviton degrees of freedom is given by $\omega = 2/q^2$. This leads us to deduce a cosmic no hair theorem for this class of models. In particular, we find that all initially expanding, spatially homogeneous Bianchi type I-VIII models will isotropize into the future when $q^2 < 2/3$. This implies that there is an open set of Bianchi models for which the scaling solution (3) is an attractor at late times.

In Sec. III, we investigate the consequences of relaxing the separable ansatz. This yields insight into the nature of scalar wave propagation in the bulk. A full analysis would involve a search for inhomogeneous solutions to the field equations of action (1) and (2). A powerful way of integrating Einstein’s equations is to apply the techniques of Hamilton-Jacobi theory to general relativity. A systematic and nonlinear scheme for solving the Hamilton-Jacobi equation has been developed by Salopek and collaborators by employing a spatial gradient expansion [15–17]. We employ this method in a five-dimensional context to solve the evolution equation for the four-metric to third-order in spatial gradients. When the dilaton field is homogeneous on the world volume, $\varphi = \varphi(y)$, it is found that the separable solution represents the attractor of the system when $q^2 < 2/3$. Combining the results of Secs. II and III therefore provides strong evidence that the scaling solution (3) represents an attractor for $q^2 < 2/3$. Finally, we conclude in Sec. IV by placing our results within the context of the holographic approach to braneworld cosmology motivated by the AdS/CFT correspondence.

II. SEPARABLE BRANEWORLDS

A. Field equations

The bulk field equations derived by extremizing action (1) and (2) are given by

$$\begin{aligned} \hat{G}_{AB} = & \frac{1}{2}\hat{\nabla}_A\varphi\hat{\nabla}_B\varphi + \frac{1}{2}e^{-b\varphi}\hat{\nabla}_A\sigma\hat{\nabla}_B\sigma \\ & - \hat{g}_{AB}\left[\frac{1}{4}(\hat{\nabla}\varphi)^2 + \frac{1}{4}e^{-b\varphi}(\hat{\nabla}\sigma)^2 + \frac{V_0}{2}e^{-q\varphi}\right] \\ & + \hat{g}_{\mu A}\hat{g}_{\nu B}e^{-q\varphi/2}\left[\mu_1\delta(y)g_{(1)}^{\mu\nu}\sqrt{\frac{g_{(1)}}{\hat{g}}}\right. \\ & \left.+ \mu_2\delta(y-\pi)g_{(2)}^{\mu\nu}\sqrt{\frac{g_{(2)}}{\hat{g}}}\right] \end{aligned} \quad (4)$$

$$\begin{aligned} \hat{\square}\varphi = & -\frac{b}{2}e^{-b\varphi}(\hat{\nabla}\sigma)^2 - qV_0e^{-q\varphi} + qe^{-q\varphi/2} \\ & \times \left[\mu_1\delta(y)\sqrt{\frac{g_{(1)}}{\hat{g}}} + \mu_2\delta(y-\pi)\sqrt{\frac{g_{(2)}}{\hat{g}}}\right] \end{aligned} \quad (5)$$

$$\hat{\nabla}_A(e^{-b\varphi}\sqrt{-\hat{g}}\hat{g}^{AB}\hat{\nabla}_B\sigma) = 0. \quad (6)$$

In this section, we assume a bulk metric of the general form

$$d\hat{s}^2 = H^m(f_{\mu\nu}dx^\mu dx^\nu + e^{2\beta}dy^2), \quad (7)$$

where the world-volume metric is represented by $f_{\mu\nu} = f_{\mu\nu}(x^\rho)$, $H = H(y)$ denotes the warp factor, $\beta = \beta(x)$ may be interpreted in four dimensions as a “radion” field and $m \equiv 4/(3q^2 - 2)$. The components of the five-dimensional Ricci tensor compatible with the metric (7) are then given by

$$\begin{aligned} \hat{R}_{\mu\nu} = & R_{\mu\nu} - \nabla_{\mu\nu}\beta - \nabla_\mu\beta\nabla_\nu\beta \\ & + \frac{m}{2}e^{-2\beta}\left[\left(1 - \frac{3m}{2}\right)\frac{H'^2}{H^2} - \frac{H''}{H}\right]f_{\mu\nu}, \end{aligned} \quad (8)$$

$$\hat{R}_{\mu y} = \frac{3m}{2}\frac{H'}{H}\nabla_\mu\beta, \quad (9)$$

$$\begin{aligned} G_{\mu\nu} - \nabla_{\mu\nu}\beta + f_{\mu\nu}\square\beta - \left(1 + \frac{2}{q^2}\right)\nabla_\mu\beta\nabla_\nu\beta + \left(1 + \frac{1}{q^2}\right)f_{\mu\nu}(\nabla\beta)^2 \\ = -\frac{1}{4}\sigma_0^2e^{-[2+(b/q)]\beta}f_{\mu\nu} + f_{\mu\nu}e^{-2\beta}\left[-\frac{6}{3q^2-2}\frac{H''}{H} + \frac{3(3q^2-8)}{(3q^2-2)^2}\frac{H'^2}{H^2} - \frac{V_0}{2H^2} + \frac{1}{H}[\mu_1\delta(y) + \mu_2\delta(y-\pi)]\right] \end{aligned} \quad (13)$$

and

$$\begin{aligned} \frac{R}{2} - \frac{1}{q^2}(\nabla\beta)^2 = & -\frac{\sigma_0^2}{4}e^{-[2+(2b/q)]\beta} + e^{-2\beta}\left[\frac{3(8-3q^2)}{(3q^2-2)^2}\right. \\ & \left.\times \frac{H'^2}{H^2} + \frac{V_0}{2H^2}\right], \end{aligned} \quad (14)$$

respectively, whereas the scalar field Eq. (5) takes the form

$$\begin{aligned} e^{-\beta}\square e^\beta + \frac{bq}{4}\sigma_0^2e^{-[2+(2b/q)]\beta} \\ = -\frac{q^2}{2}e^{-2\beta}\left[\frac{6}{3q^2-2}\frac{H''}{H} + \frac{6(8-3q^2)}{(3q^2-2)^2}\frac{H'^2}{H^2} \right. \\ \left. + \frac{V_0}{H^2} - \frac{1}{H}(\mu_1\delta(y) + \mu_2\delta(y-\pi))\right]. \end{aligned} \quad (15)$$

A crucial property of Eqs. (13)–(15) is that the terms contained within the square brackets are *independent* of the world-volume coordinates, x^μ . We therefore define three separation constants, c_i :

$$\begin{aligned} -\frac{6}{3q^2-2}\frac{H''}{H} + \frac{3(3q^2-8)}{(3q^2-2)^2}\frac{H'^2}{H^2} - \frac{V_0}{2H^2} + \frac{1}{H}[\mu_1\delta(y) \\ + \mu_2\delta(y-\pi)] \equiv c_1, \end{aligned} \quad (16)$$

$$\frac{3(8-3q^2)}{(3q^2-2)^2}\frac{H'^2}{H^2} + \frac{V_0}{2H^2} \equiv c_2, \quad (17)$$

$$\hat{R}_{yy} = -e^{2\beta}[\square\beta + (\nabla\beta)^2] + 2m\left(\frac{H'^2}{H^2} - \frac{H''}{H}\right), \quad (10)$$

where a prime denotes differentiation with respect to y .

B. Separable branes

A separable solution between the world volume and bulk coordinates can be found by assuming the *ansatz* $\varphi = \varphi_1(x) + \varphi_2(y)$ and $\sigma = \sigma(y)$ and specifying $b = 2/q$. The (μy) -components of the Einstein field Eqs. (4) are then solved directly by

$$\varphi_1 = \frac{2}{q}\beta, \quad \varphi_2 = \frac{6q}{3q^2-2}\ln H, \quad (11)$$

and the axion field Eq. (6) admits the first integral

$$\sigma' = \sigma_0 H^{6/(3q^2-2)} \quad (12)$$

for an arbitrary constant σ_0 .

The $(\mu\nu)$ - and (yy) -components of the field Eqs. (4) then reduce to

$$\begin{aligned} -\frac{q^2}{2}\left[\frac{6}{3q^2-2}\frac{H''}{H} + \frac{6(8-3q^2)}{(3q^2-2)^2}\frac{H'^2}{H^2} + \frac{V_0}{H^2} - \frac{1}{H}[\mu_1\delta(y) \right. \\ \left. + \mu_2\delta(y-\pi)]\right] \equiv c_3. \end{aligned} \quad (18)$$

This implies that Eqs. (13)–(15) simplify to

$$\begin{aligned} G_{\mu\nu} - \nabla_{\mu\nu}\beta + f_{\mu\nu}\square\beta - \left(1 + \frac{2}{q^2}\right)\nabla_\mu\beta\nabla_\nu\beta \\ + \left(1 + \frac{1}{q^2}\right)f_{\mu\nu}(\nabla\beta)^2 = -\frac{1}{4}\sigma_0^2e^{-[2+(b/q)]\beta}f_{\mu\nu} \\ + c_1f_{\mu\nu}e^{-2\beta} \end{aligned} \quad (19)$$

$$\frac{R}{2} - \frac{1}{q^2}(\nabla\beta)^2 = -\frac{\sigma_0^2}{4}e^{-[2+(2b/q)]\beta} + c_2e^{-2\beta}, \quad (20)$$

$$\square\beta + (\nabla\beta)^2 = -\frac{bq}{4}\sigma_0^2e^{-[2+(2b/q)]\beta} + c_3e^{-2\beta}. \quad (21)$$

However, the constants c_i are not independent. Substituting the trace of Eq. (19) into Eq. (20) and comparing with Eq. (21) implies that

$$3c_3 = 4c_1 + 2c_2 \quad (22)$$

and subtracting Eq. (16) from Eq. (18) and comparing with Eq. (17) implies that

$$c_1 = c_2 + \frac{2}{q^2} c_3. \quad (23)$$

Equations (22) and (23) then imply that

$$6c_1 = \left(3 + \frac{4}{q^2}\right)c_3. \quad (24)$$

It may now be verified directly that Eqs. (19)–(21) follow by extremizing the effective four-dimensional action

$$S_4 = \int d^4x \sqrt{-f} e^\beta \left[R^{(f)} - \frac{2}{q^2} (\nabla \beta)^2 - V \right] \quad (25)$$

$$V(\beta) \equiv -2c_1 e^{-2\beta} + \frac{\sigma_0^2}{2} e^{-[2+(4/q^2)]\beta}, \quad (26)$$

where $R^{(f)}$ is the Ricci curvature scalar of the metric $f_{\mu\nu}$ and $f \equiv \det f_{\mu\nu}$. The action (25) represents an effective Brans-Dicke scalar-tensor theory of gravity with a constant dilaton-graviton coupling parameter given by $\omega = 2/q^2$. It should be emphasized that no *a priori* assumption has been made regarding the form of the world-volume metric, $f_{\mu\nu}$. Thus, we have found a general class of warped, five-dimensional geometries of the form (7), where the world-volume metric satisfies the field equations derived from the effective action (25). Models of this type, both with and without an axion field, but with vanishing separation constants $c_i = 0$, have been analyzed previously in a number of settings [10,18,19].

C. Warp factor

It only remains to solve Eqs. (16)–(18) for the warp factor $H(y)$. However, these equations are independent of the world-volume coordinates, so the warp factor takes an identical form to that for the spatially flat FRW world-volume. Its precise form is determined by the values of the parameters $\{q, V_0, c_1\}$ and, since our interest here is in the nature of the world-volume metric, we refer the reader to Refs. [12,13] for further details. As an example, however, consider the case of a two-brane scenario supported by a negative bulk potential, $V_0 < 0$, where $q^2 < 2/3$ and $c_1 < 0$. It follows from Eqs. (16)–(18) that

$$H = \sqrt{\frac{|V_0|}{2c_2}} \sinh(D|y - y_0|), \quad D \equiv \sqrt{\frac{c_2(3q^2 - 2)^2}{3(8 - 3q^2)}}, \quad (27)$$

where the integration constant y_0 is specified such that the branes are located at $y = (0, \pi)$, i.e., $H(0) = H(\pi) = 1$. This implies that the brane tensions are given by

$$\mu_1 = -\sqrt{\frac{24(2c_2 + |V_0|)}{8 - 3q^2}}, \quad \mu_1 = -\mu_2, \quad (28)$$

where the second relation follows as a consequence of the Z_2 reflection symmetry.

D. Cosmic no hair on the world volume

We are now able to deduce a cosmic no hair theorem for this class of braneworlds when the world-volume represents a spatially homogeneous but anisotropic Bianchi spacetime. (A Bianchi metric admits three-dimensional, spacelike hypersurfaces on which a three-parameter Lie group of isometries acts simply transitively). To proceed, we note that action (25) may be transformed into the Einstein-Hilbert action for a minimally coupled, self-interacting scalar field by the conformal transformation

$$\tilde{f}_{\mu\nu} = \Omega^2 f_{\mu\nu}, \quad \Omega^2 \equiv e^\beta \quad (29)$$

and field redefinition

$$\chi \equiv \sqrt{3 + \frac{4}{q^2}} \beta. \quad (30)$$

It follows that

$$\tilde{S} = \int d^4x \sqrt{-\tilde{f}} \left[\tilde{R} - \frac{1}{2} (\tilde{\nabla} \chi)^2 - \tilde{V}(\chi) \right], \quad (31)$$

where

$$\tilde{V} = -2c_1 e^{-\lambda \chi} + \frac{\sigma_0^2}{2} e^{-3\chi/\lambda}, \quad \lambda \equiv \frac{3|q|}{\sqrt{4 + 3q^2}}. \quad (32)$$

The effective potential (32) contains two contributions, one from the axion field and the other from the nontrivial separation constant c_1 . When $c_1 < 0$, the potential is positive-definite and contains no turning points. This implies that $\chi \rightarrow +\infty$ at late-times for any ever-expanding cosmology. (For the case where $c_1 > 0$, the potential exhibits a minimum but its value is negative at this point). Since the contribution to the potential sourced by the axion field is steeper, the self-interactions of the field become dominated at late-times by the contribution arising from the separation constant. Now, the conformal transformation (29) and (30) is well-defined for the class of spatially homogeneous and anisotropic Bianchi metrics and, moreover, the two metrics in Eq. (29) correspond to the same Bianchi type if the radion field β is constant on the surfaces of homogeneity, i.e., the Bianchi type is invariant under the conformal transformation. This implies that the known results from four-dimensional general relativity coupled to an exponential potential can be carried over directly to this braneworld scenario [2]. We are therefore led to the following cosmic no hair theorem for braneworlds with a separable metric of the form (7) supported by a bulk exponential potential: for $q^2 < 2/3$, all initially expanding, spatially homogeneous Bianchi world-volumes (except for a subclass of Bianchi type IX models that recollapse) isotropize in the future toward the power-law inflationary, spatially flat FRW metric $ds^2 = -d\tau^2 + \tau^{4/(3q^2)} \delta_{ij} dx^i dx^j$.

(The form of the world-volume metric follows after conformally transforming back to the original frame (25).)

Thus, the scaling solution (3) represents a late-time attractor for spatially homogeneous models when the separable ansatz (7) applies and $q^2 < 2/3$. In the following section, we investigate the implications of relaxing this assumption on the form of the bulk metric.

III. INHOMOGENEITIES IN THE BULK AND THE HAMILTON-JACOBI FORMALISM

In the previous section we deduced a cosmic no hair result for braneworlds satisfying the separable ansatz (7) with a spatially homogeneous world-volume. We now wish to provide evidence that the scaling metric (3) also represents an attractor under more general inhomogeneous settings. This involves a study of the five-dimensional bulk solutions when the separable ansatz is relaxed. A powerful framework for solving the Einstein field equations sourced by a self-interacting scalar field is provided by the Hamilton-Jacobi (HJ) formalism of general relativity. We briefly review this formalism in a five-dimensional context in the following subsection and then proceed to investigate the evolution of the bulk metric.

A. Hamilton-Jacobi equation

We consider the five-dimensional sector of action (1) with vanishing axion field and investigate bulk metrics in the Arnowitt-Deser-Misner (ADM) form [20]

$$ds^2 = (N^2 + \gamma_{\mu\nu} N^\mu N^\nu) dy^2 + 2N_\mu dy dx^\mu + \gamma_{\mu\nu} dx^\mu dx^\nu, \quad (33)$$

where $\gamma_{\mu\nu} = \gamma_{\mu\nu}(x^\rho, y)$. Rewriting action (1) in a Hamiltonian form and varying with respect to the lapse and shift functions, N and N^μ , yields the Hamiltonian and momentum constraints, whereas the equations of motion for $\gamma_{\mu\nu}$ and φ follow by varying the action with respect to their conjugate momenta, $\pi^{\mu\nu}$ and π^φ :

$$\partial_y \gamma_{\mu\nu} - \nabla_\nu N_\mu - \nabla_\mu N_\nu = -2 \frac{N}{\sqrt{-\gamma}} \pi^{\lambda\rho} \times \left(\gamma_{\mu\rho} \gamma_{\nu\lambda} - \frac{1}{3} \gamma_{\mu\nu} \gamma_{\lambda\rho} \right), \quad (34)$$

$$\partial_y \varphi - N^\mu \nabla_\mu \varphi = - \frac{N}{\sqrt{-\gamma}} \pi^\varphi. \quad (35)$$

The evolution equations for the momenta are automatically solved [20] by defining

$$\pi^{\mu\nu} = \frac{\delta S}{\delta \gamma_{\mu\nu}}, \quad \pi^\varphi = \frac{\delta S}{\delta \varphi} \quad (36)$$

and requiring that these satisfy the Hamiltonian and momentum constraints when the equations of motion (34) and (35) are satisfied. The functional S represents the generat-

ing functional of the HJ equation. The momentum constraint implies that this functional should be diffeomorphism invariant [21] and the HJ equation then represents the Hamiltonian constraint. This is a hyperbolic, functional partial differential equation for S and may be expressed in the form

$$\{S, S\} = \mathcal{L}_4, \quad (37)$$

where we have defined the bracket:

$$\{S, S\} \equiv - \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{\mu\nu}} \frac{\delta S}{\delta \gamma_{\lambda\rho}} \left(\gamma_{\mu\rho} \gamma_{\nu\lambda} - \frac{1}{3} \gamma_{\mu\nu} \gamma_{\lambda\rho} \right) - \frac{1}{2\sqrt{-\gamma}} \left(\frac{\delta S}{\delta \varphi} \right)^2 \quad (38)$$

and

$$\mathcal{L}_4 \equiv \sqrt{-\gamma} R(\gamma) - \frac{1}{2} \sqrt{-\gamma} \gamma^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - \sqrt{-\gamma} V(\varphi). \quad (39)$$

Equations (34)–(36) yield the full set of evolution equations given a solution to the HJ Eq. (37). The key idea of the spatial gradient expansion method [15–17] is to derive an order-by-order solution of the HJ equation by expanding the generating functional in a series $S = \sum_{n=0}^{\infty} S^{(2n)}$, where $2n$ represents the number of spatial gradients in $S^{(2n)}$. The HJ equation is also expanded in spatial gradients, $\mathcal{H} = \sum_{n=0}^{\infty} \mathcal{H}^{(2n)} = 0$, and is then required to vanish at each order in n . For $n = (0, 2, 4)$, this leads to the constraints:

$$\{S^{(0)}, S^{(0)}\} = -\sqrt{-\gamma} V(\varphi), \quad (40)$$

$$2\{S^{(0)}, S^{(2)}\} = \sqrt{-\gamma} R - \frac{1}{2} \sqrt{-\gamma} (\nabla \varphi)^2, \quad (41)$$

$$2\{S^{(0)}, S^{(4)}\} + \{S^{(2)}, S^{(2)}\} = 0. \quad (42)$$

The zero-order Eq. (40) is solved by [15]

$$S^{(0)} = - \int d^4x \sqrt{-\gamma} W(\varphi), \quad (43)$$

where the function $W(\varphi)$ is a solution to the ordinary differential equation (ODE):

$$\frac{1}{2} \left(\frac{dW}{d\varphi} \right)^2 - \frac{1}{3} W^2 = V. \quad (44)$$

Each solution to Eq. (44) is characterized by the value of a single parameter field, $\tilde{\varphi}$, such that $W = W(\varphi, \tilde{\varphi})$. Differentiating (44) with respect to $\tilde{\varphi}$ then implies that the solution to Eq. (44) can be expressed as [16]

$$W = \frac{3}{2} \frac{\partial W}{\partial \varphi} \frac{\partial}{\partial \varphi} \ln \left(\frac{\partial W}{\partial \tilde{\varphi}} \right). \quad (45)$$

The second-order Eq. (41) is solved by substituting the ansatz

$$S^{(2)} = \int d^4x \sqrt{-\gamma} \left[J(\varphi)R - \frac{1}{2}K(\varphi)(\nabla\varphi)^2 \right] \quad (46)$$

for the functions $J(\varphi)$ and $K(\varphi)$ and requiring the coefficients of the terms involving R , $\square\varphi$, and $(\nabla\varphi)^2$ to vanish identically. This results in three coupled ODEs [16]:

$$\frac{dW}{d\varphi} \frac{dJ}{d\varphi} - \frac{1}{3}WJ = 1, \quad (47)$$

$$W \frac{dJ}{d\varphi} + K \frac{dW}{d\varphi} = 0, \quad (48)$$

$$2W \frac{d^2J}{d\varphi^2} + \frac{dK}{d\varphi} \frac{dW}{d\varphi} + \frac{1}{3}WK = -1. \quad (49)$$

Equations (47)–(49) are not independent and Eq. (49) follows after differentiating Eqs. (47) and (48) with respect to φ . Moreover, Eq. (47) may be solved in terms of an integrating factor once a solution to Eq. (44) has been found. It follows, after substitution of the solution (45) into Eq. (47), that

$$J = \left(\frac{\partial W}{\partial \tilde{\varphi}} \right)^{1/2} \int d\varphi \left(\frac{\partial W}{\partial \varphi} \right)^{-1} \left(\frac{\partial W}{\partial \tilde{\varphi}} \right)^{-1/2}. \quad (50)$$

In principle, therefore, the HJ equation can be solved perturbatively once a solution to the zero-order Eq. (44) has been found. We consider the case of an exponential potential in the following subsections.

B. Solution for a bulk exponential potential

In general, Eq. (44) may be viewed as a first-order ODE with a “time” variable φ [22]. By defining a new dependent variable²

$$Y \equiv -\sqrt{\frac{2}{3}} \frac{W}{dW/d\varphi}, \quad \frac{1}{Y^2} = 1 + 3 \frac{V}{W^2}, \quad (51)$$

this ODE may be expressed in the form of a first-order Abel equation:

$$\sqrt{\frac{3}{2}} \frac{dY}{d\varphi} = \sqrt{\frac{3}{8}} \frac{d \ln V}{d\varphi} Y^3 + Y^2 - \sqrt{\frac{3}{8}} \frac{d \ln V}{d\varphi} Y - 1. \quad (52)$$

For a scalar field with a negative exponential potential, $V = V_0 \exp(-q\varphi)$ with $V_0 < 0$ and $q > 0$, it proves convenient to define a further variable $Y \equiv \Theta^{-1/2}$. Since Θ is bounded such that $0 \leq \Theta^2 \leq 1$, this implies that Eq. (52) can then be written as a one-dimensional nonlinear dynamical system:

$$\frac{d\Theta}{d\varphi} = \sqrt{\frac{8}{3}} (\Theta - 1)(\sqrt{\Theta} - s), \quad (53)$$

²We assume implicitly and without loss of generality that $dW/d\varphi < 0$.

where $s \equiv \sqrt{3/8}q$. The equilibrium points for this system are at $\Theta_{\text{eqm}} = s^2$ and $\Theta_{\text{eqm}} = 1$, respectively. A stability analysis in the neighborhood of these points implies that the former is stable for $q < \sqrt{8/3}$ and unstable for $q > \sqrt{8/3}$, whereas the latter equilibrium point is unstable for $q < \sqrt{8/3}$ and stable for $q > \sqrt{8/3}$. Since Θ is bounded, these points represent the global attractor and repeller in the phase space. The point $\Theta_{\text{eqm}} = 1$ corresponds to the limit where the potential energy of the field is dynamically negligible, whereas $\Theta_{\text{eqm}} = s^2$ represents scaling behavior. We therefore focus in the remainder of this section on the region of parameter space where $q < \sqrt{8/3}$. Moreover, the general solution to Eq. (52) is given by

$$\frac{(1+Y)^{s-1}(1-Y)^{s+1}}{(1-sY)^{2s}} = \exp \left[\sqrt{\frac{8}{3}} (1-s^2)(\varphi - \varphi_m) \right], \quad (54)$$

where φ_m is an integration constant, and it follows that the stable equilibrium point corresponds to $\varphi \rightarrow +\infty$. (This will be important when calculating the asymptotic form of the third-order metric $\gamma_{\mu\nu}^{(3)}$ in Sec. III D.)

The attractor solution to the zero-order HJ equation now follows immediately from Eq. (51):

$$W(\varphi) = W_0 e^{-q\varphi/2}, \quad W_0 = \pm \sqrt{\frac{24V_0}{3q^2 - 8}}. \quad (55)$$

Equations (47) and (48) are then solved by

$$K(\varphi) = J(\varphi), \quad J(\varphi) = J_0 e^{q\varphi/2}, \quad (56)$$

$$W_0 J_0 = -\frac{12}{3q^2 + 4},$$

and the generating functional to second-order in spatial gradients is therefore given by

$$S^{(0)} + S^{(2)} = \int d^4x \sqrt{-\gamma} \left[J_0 e^{q\varphi/2} \left(R - \frac{1}{2}(\nabla\varphi)^2 \right) - W_0 e^{-q\varphi/2} \right]. \quad (57)$$

Modulo trivial rescalings and a field redefinition $\varphi \rightarrow 2\beta/q$, the coupling between the scalar and tensor degrees of freedom in Eq. (57) is *precisely* the same as the coupling in the effective four-dimensional action (25), i.e., it is of the Brans-Dicke form where $\omega = 2/q^2$. The qualitative forms of the potential terms are also identical.

C. Fourth-order Hamiltonian

We now solve the fourth-order Eq. (42). Substitution of the variations of $S^{(0)}$ yields the first-order functional differential equation

$$W \frac{\delta S^{(4)}}{\delta \gamma_{\mu\nu}} \gamma_{\mu\nu} - 3 \frac{dW}{d\varphi} \frac{\delta S^{(4)}}{\delta \varphi} = 3\{S^{(2)}, S^{(2)}\} \quad (58)$$

and this equation can be solved by employing the conformal transformation technique of Ref. [17]. This involves defining the set of new variables:

$$\gamma_{\mu\nu} \equiv \Omega^2(u) k_{\mu\nu}, \quad \frac{\partial \Omega}{\partial u} = \frac{W}{2} \Omega, \quad (59)$$

$$u \equiv -\frac{1}{3} \int d\varphi \left(\frac{dW}{d\varphi} \right)^{-1},$$

and transforming Eq. (58) into the form

$$\frac{\delta S^{(4)}}{\delta u} = 3\{S^{(2)}, S^{(2)}\}. \quad (60)$$

Equation (60) then admits a solution in terms of the line integral

$$S^{(4)} = 3 \int_0^u du' \int d^4x \mathcal{R}^{(4)}[u'(x), k_{\mu\nu}(x)], \quad (61)$$

where $\mathcal{R}^{(4)} \equiv \{S^{(2)}, S^{(2)}\}$ is to be viewed as a functional of $u'(x)$ and the conformal metric $k_{\mu\nu}$. The integral (61) may be evaluated by choosing a straight line path such that [17]

$$u'(x) = ru(x), \quad du'(x) = u(x)dr, \quad (62)$$

where r is a real parameter taking values in the range $0 \leq r \leq 1$. Each term in $\mathcal{R}^{(4)}$ depends quadratically on r and, consequently, the integral over u reduces to performing the trivial integration $\int_0^1 r^2 dr$. The solution to Eq. (42) is therefore given by

$$S^{(4)} = \int d^4x u \{S^{(2)}, S^{(2)}\}. \quad (63)$$

The explicit form of the fourth-order contribution to the generating functional then follows after substitution of the second-order contribution $S^{(2)}$. In the case of the attractor solution (55) and (56) for the exponential potential, it follows from Eq. (59) that $u = 4/(3q^2W)$. Hence, we need only substitute the second-order term, Eq. (57), into the integral (63) and integrate by parts where appropriate. We find, after some algebra, that

$$\begin{aligned} S^{(4)} = & -\frac{4J_0^2}{3q^2W_0} \int d^4x \sqrt{-\gamma} e^{3q\varphi/2} \left[R_{\mu\nu} R^{\mu\nu} - \left(\frac{1}{3} - \frac{q^2}{8} \right) R^2 \right. \\ & + \frac{1}{2} (\square\varphi)^2 + \frac{q}{2} R \square\varphi + \left(\frac{1}{3} - \frac{3}{8} q^2 \right) R (\nabla\varphi)^2 \\ & - \left(1 - \frac{3q^2}{4} \right) R_{\mu\nu} \nabla^\mu \varphi \nabla^\nu \varphi - \frac{q}{4} \left(1 - \frac{3q^2}{4} \right) \square\varphi (\nabla\varphi)^2 \\ & \left. + \left(\frac{1}{6} - \frac{7q^2}{32} + \frac{3q^4}{32} \right) (\nabla\varphi)^4 \right]. \quad (64) \end{aligned}$$

D. Evolution of the world-volume metric

We are now able to evaluate the evolution of the metric $\gamma_{\mu\nu}$ up to third-order in derivatives. To proceed, we specify the gauge such that the shift function vanishes, $N^\mu = 0$, and further assume that the scalar field is constant on surfaces of constant y , i.e., $\varphi = \varphi(y)$. Moreover, we identify the value of the scalar field as the “time” parameter representing evolution in the fifth dimension. This is equivalent to choosing the lapse function to be

$$\frac{1}{N} = \frac{\partial W}{\partial \varphi} - \frac{\partial J}{\partial \varphi} R, \quad (65)$$

as follows directly from Eq. (35).

The evolution of the metric $\gamma_{\mu\nu}$ to first-order is determined by truncating the generating functional at the lowest-order term, $S = S^{(0)}$, and setting the lapse $N^{-1} = \partial W / \partial \varphi$ in Eq. (34) [16]. Integrating (34) then yields

$$\gamma_{\mu\nu}^{(1)} = \left(\frac{\partial W}{\partial \tilde{\varphi}} \right)^{-1/2} h_{\mu\nu}(x^\rho), \quad (66)$$

where we have employed expression (45) and the conformal metric, $h_{\mu\nu}(x)$, is independent of the fifth coordinate.

The evolution of the metric to third-order, on the other hand, is determined by truncating the expansion of the generating functional at $n = 1$. After substituting the variation of $S^{(2)}$, as determined from Eq. (46), into the evolution Eq. (34), it follows that

$$\frac{\partial_y \gamma_{\mu\nu}}{N} = -\frac{1}{3} W \gamma_{\mu\nu} + 2J \left(R_{\mu\nu} - \frac{1}{6} R \gamma_{\mu\nu} \right), \quad (67)$$

where the second term on the right hand side is evaluated with the first-order (long-wavelength) metric (66) and the other two terms contain contributions from the first- and third-order metrics. The third-order metric is determined by integrating (67) after substitution of Eq. (65), where the substitution is done in such a way that only terms up to second-order in spatial gradients are retained and first-order results are substituted into second-order terms [16]. It is found that

$$\begin{aligned} \gamma_{\mu\nu}^{(3)}(x^\rho, \varphi) = & \left(\frac{\partial W}{\partial \tilde{\varphi}} \right)^{-1/2} h_{\mu\nu} - \frac{1}{3} \left(\frac{\partial W}{\partial \tilde{\varphi}} \right)^{-1/2} \int d\varphi' W \frac{\partial J}{\partial \varphi'} \\ & \times \left(\frac{\partial W}{\partial \varphi'} \right)^{-2} \left(\frac{\partial W}{\partial \tilde{\varphi}} \right)^{1/2} R^{(h)} h_{\mu\nu} + 2 \left(\frac{\partial W}{\partial \tilde{\varphi}} \right)^{-1/2} \\ & \times \int d\varphi' J \left(\frac{\partial W}{\partial \varphi'} \right)^{-1} \left(\frac{\partial W}{\partial \tilde{\varphi}} \right)^{1/2} \left[R_{\mu\nu}^{(h)} \right. \\ & \left. - \frac{1}{6} R^{(h)} h_{\mu\nu} \right], \quad (68) \end{aligned}$$

where $R_{\mu\nu}^{(h)}$ and $R^{(h)}$ are the Ricci tensor and curvature scalar, respectively, of the conformal metric $h_{\mu\nu}$.

It follows from Eq. (68) that the evolution of the metric at this order is determined, at least in principle, once the zero-order HJ Eq. (44) has been solved. To determine the

asymptotic behavior of the world-volume metric, therefore, we may substitute the attractor solution (55) into Eqs. (45) and (50). This is equivalent to substituting $\partial W/\partial \tilde{\varphi} = e^{-(4/3q)\varphi}$ into Eqs. (66) and (68) and we deduce that

$$\gamma_{\mu\nu}^{(1)} = e^{(2/3q)\varphi} k_{\mu\nu}, \quad (69)$$

$$\gamma_{\mu\nu}^{(3)} = e^{(2/3q)\varphi} k_{\mu\nu} - \frac{12J_0}{W_0(3q^2 - 2)} e^{q\varphi} R_{\mu\nu}^{(k)}, \quad (70)$$

where $k_{\mu\nu}$ is directly proportional to $h_{\mu\nu}$. Hence, since $\varphi \rightarrow \infty$ corresponds to the attractor in this model when $q^2 < 8/3$, we conclude that the first-order term increases *more rapidly* than the third-order term for $q^2 < 2/3$. As a result, the metric approaches the first-order separable metric (69) for this region of parameter space. This is precisely the upper limit on the value of the coupling parameter for which the cosmic no hair theorem of Sec. II applies. Moreover, comparison with the exact braneworld (7) and (11) implies that the world-volume sector of the bulk solution (7) can be expressed in exactly the same form, $\gamma_{\mu\nu} = e^{(2/3q)\varphi} f_{\mu\nu}(x)$, as that of the first-order metric (69).

IV. DISCUSSION AND A NOTE ON HOLOGRAPHY

In this paper, we have focused on various aspects of braneworld cosmology where the bulk gravitational action contains a scalar dilaton field with an exponential self-interaction potential with coupling parameter q . A massless axion field coupled to the dilaton was also included in the bulk action. We have found a general class of braneworlds, where the world-volume metric represents any solution to a four-dimensional scalar-tensor theory of gravity where the coupling between the spin-0 and spin-2 fields takes the value $\omega = 2/q^2$. The axion field generates a potential for the dilaton in four dimensions. This generalizes the results of [11–13] to an arbitrary world-volume metric in the presence of a bulk axion field and extends the results of [18,19] to the case where the four-dimensional dilaton has a nontrivial potential.

We have argued, from both the world-volume and bulk perspectives, that the spatially flat FRW scaling solution (3) represents an asymptotic attractor in a wide variety of settings when the constraint $q^2 < 2/3$ is satisfied. Specifically, we have derived a “cosmic no hair” theorem for the class of spatially homogeneous Bianchi world-volumes. We also applied the Hamilton-Jacobi framework to five-dimensional general relativity up to third-order in metric derivatives and found that when $q^2 < 2/3$, the first-order (separable) bulk metric dominates the third-order contributions as the attractor is approached. Moreover, it is striking that the zero- and second-order contributions to the generating functional of the HJ equation take the same form as the effective Brans-Dicke action that determines the world-volume metric $f_{\mu\nu}$ in the separable bulk solu-

tion. We have also presented the first derivation of the fourth-order contribution to the generating functional for five-dimensional Einstein gravity coupled to an exponential scalar field potential.

An alternative approach to solving the bulk Einstein equations is to employ a gradient (low-energy) expansion technique directly at the level of the field equations [23,24]. The two-brane scenario with a bulk exponential potential was recently studied in this context by Leeper *et al.* [14], who found an approximate solution to the field equations where a perturbation in the position of one of the branes induces a perturbation in the bulk metric away from the separable ansatz (3). It was found that the system is stable to such a perturbation and that the bulk rapidly tends to its unperturbed form. Such an analysis differs from that of the present work and was restricted to first-order perturbations and it would clearly be of interest to extend the analysis to higher-order.

Although our primary interest has focused on bulk gravitational issues, our work also overlaps with recent developments in holographic approaches to cosmology and we now conclude with a discussion on these issues. The AdS/CFT correspondence [25] states that gravity on $(d+1)$ -dimensional anti-de Sitter (AdS) space admits a dual description in terms of a conformal field theory (CFT) on the d -dimensional boundary. (For a review, see, [26]). Within this context, the radial bulk coordinate is identified as a renormalization group (RG) flow parameter (energy scale) of the dual field theory, such that the evolution of the bulk fields along the radial direction induces nonvanishing β -functions in the dual theory [27–29]. As is well known, however, the supergravity action diverges when the AdS boundary is taken to infinity. de Boer, Verlinde, and Verlinde [29] have advocated a method of handling such divergences within a holographic RG approach based on the HJ formalism. The effective action for the gauge theory, Γ , is related to the bulk classical action, S , by $S = S_{\text{loc}} + \Gamma$, where S is evaluated on a solution to the bulk field equations (with appropriate boundary conditions), S_{loc} represents the divergent terms (that are no higher than second-order in derivatives) and Γ contains the higher-order, nonlocal contributions. A primary motivation for such an approach is that Eq. (42), determining the fourth-order contribution to the generating functional of the HJ equation, takes the form of a “Callan-Symanzik” equation [29]:

$$\gamma^{\mu\nu} \frac{\delta \Gamma}{\delta \gamma^{\mu\nu}} = \beta(\varphi) \frac{\delta \Gamma}{\delta \varphi} + \frac{3}{W} \{S_{\text{loc}}, S_{\text{loc}}\}, \quad (71)$$

when we identify $S^{(4)} = \Gamma$, $S^{(2)} = S_{\text{loc}}$, and $\beta \equiv 3d \ln W/d\varphi$ with the β -function of the dual theory. The left-hand side of Eq. (71) is precisely the conformal anomaly of the gauge theory [30–33].

Within the context of the braneworld paradigm, this provides strong motivation for interpreting the braneworld

as a cutoff, strongly coupled conformal gauge theory coupled to four-dimensional gravity with a dual action $\tilde{S} = S_{\text{loc}} + \Gamma + S_{\text{brane}}$, where S_{brane} represents the brane [30,31,34]. Thus, the dual effective action in the presence of a bulk dilaton scalar field would be determined in this context from Eqs. (40)–(48).

The bulk solutions found in Sec. II represent domain-wall backgrounds, but these are not asymptotically AdS since the exponential potential does not contain a global minimum. Nonetheless, the above discussion should apply to any model where the bulk potential has an approximately exponential form over some finite range of scalar field values and, in this case, the fourth-order contribution to the HJ generating functional that we have derived in Eq. (64) may then be identified as the effective action for the conformal anomaly in this regime. It would clearly be of interest to establish the necessary conditions for inflation to arise from such a contribution, since its ultraviolet nature implies that it may play a dominant role in the very early universe. On the other hand, we have found that the low-energy limit of the dual effective action can isotropize the braneworld if the logarithmic derivative of the (negative) bulk potential is sufficiently flat, i.e., the β -function of the gauge theory is sufficiently small.

Finally, the bulk solutions we have investigated in the present work may also be relevant to the proposed domain-wall/quantum field theory (DW/QFT) correspondence [35], which exploits the fact that AdS space (in horospherical coordinates) represents a special case of a domain-wall background. This suggests—in view of the dualities that relate all brane backgrounds—that the AdS/CFT correspondence can be extended to that of an ordinary QFT living on the boundary of the domain wall. More specifically, the metric of AdS space in horospherical coordinates with radius of curvature ℓ is given by $ds^2 = \ell^2(du^2/u^2) + (u^2/\ell^2)\eta_{\mu\nu}dx^\mu dx^\nu$, where $\eta_{\mu\nu}$ is the metric for flat space. In general, the bulk metric we have considered as an ansatz in Eq. (7) can not be expressed in this form. However, when the separable condition (11) is satisfied, a conformal transformation

$$ds_{\text{dual}}^2 = e^{-q\varphi} d\hat{s}_5^2 \quad (72)$$

on the metric (7) results in the “dual” metric

$$ds_{\text{dual}}^2 = H^{-2}(y)(e^{-2\beta} f_{\mu\nu} dx^\mu dx^\nu + dy^2). \quad (73)$$

Comparison with the warp factor (27), for example, then implies that in the limit $y \rightarrow y_0$, the metric (73) can indeed be expressed in the horospherical AdS form (after a trivial rescaling of the coordinates) by identifying $(y - y_0) \propto u^{-1}$.

In the DW/QFT correspondence, the horospherical coordinate u is identified as the energy scale of the dual theory, with $u = \infty$ corresponding to the AdS boundary [35]. The dual frame (73) therefore provides the natural context for discussing the DW/QFT correspondence. Moreover, performing the conformal transformation (72) on the bulk action (1) and (2) results in a scalar-tensor gravity theory:

$$S_{\text{dual}} = \int d^5x \sqrt{-g} e^{-\varphi_{\text{dual}}} [R - \omega_{\text{dual}} (\nabla \varphi_{\text{dual}})^2 - V_0], \quad (74)$$

where

$$\varphi_{\text{dual}} \equiv -\frac{3q}{2}\varphi, \quad \omega_{\text{dual}} \equiv \frac{2}{9q^2}(1 - 6q^2). \quad (75)$$

It is of interest to note that the critical value we have identified for the attractor scaling solution, $q^2 = 2/3$, corresponds precisely to the coupling that arises in the dilaton-graviton sector of the string effective action, $\omega_{\text{dual}} = -1$. In conclusion, therefore, we anticipate that the class of braneworlds we have found will play a key role in developing the holographic approach to braneworld cosmology in terms of the AdS/CFT and DW/QFT correspondences.

ACKNOWLEDGMENTS

DS is supported by PPARC. We thank A. Coley, K. Koyama, E. Leeper and S. Odintsov for helpful comments and discussions.

-
- [1] R. M. Wald, *Phys. Rev. D* **28**, 2118 (1983).
 - [2] Y. Kitada and K. Maeda, *Phys. Rev. D* **45**, 1416 (1992); Y. Kitada and K. Maeda, *Classical Quantum Gravity* **10**, 703 (1993).
 - [3] L. G. Jensen and J. A. Stein-Schabes, *Phys. Rev. D* **35**, 1146 (1987); M. Heusler, *Phys. Lett. B* **253**, 33 (1991); J. Ibáñez, R. J. van den Hoogen, and A. A. Coley, *Phys. Rev. D* **51**, 928 (1995).
 - [4] J. Polchinski, *String Theory*, Vol. 1–2 (Cambridge University Press, Cambridge, 1998); C. V. Johnson, *D-Branes* (Cambridge University Press, Cambridge, 2003);

- T. Ortín, *Gravity and Strings*, (Cambridge University Press, Cambridge, 2004).
- [5] P. Brax and C. van de Bruck, *Classical Quantum Gravity* **20**, R201 (2003); R. Maartens, *Living Rev. Relativity* **7**, 7 (2004).
- [6] R. Maartens, V. Sahni, and T. D. Saini, *Phys. Rev. D* **63**, 063509 (2001); M. G. Santos, F. Vernizzi, and P. G. Ferreira, *Phys. Rev. D* **64**, 063506 (2001); A. V. Toporensky, *Classical Quantum Gravity* **18**, 2311 (2001); J. D. Barrow and S. Hervik, *Classical Quantum Gravity* **19**, 155 (2002); B. C. Paul, *Phys. Rev. D* **66**,

- 124019 (2002); M. Bruni and P. K. S. Dunsby, Phys. Rev. D **66**, 101301 (2002); J. M. Aguirregabiria, L. P. Chimento, and R. Lazkoz, Classical Quantum Gravity **21**, 823 (2004); A. A. Coley and S. Hervik, Classical Quantum Gravity **20**, 3061 (2003); P. K. S. Dunsby, N. Goheer, M. Bruni, and A. A. Coley, Phys. Rev. D **69**, 101303 (2004); A. A. Coley, Y. He, and W. C. Lim, Classical Quantum Gravity **21**, 1311 (2004); A. A. Coley, astro-ph/0504226.
- [7] C. Cadeau and E. Woolgar, Classical Quantum Gravity **18**, 527 (2001); A. Campos, R. Maartens, D. Matravers, and C. F. Sopuerta, Phys. Rev. D **68**, 103520 (2003); A. Fabbri, D. Langlois, D. A. Steer, and R. Zegers, J. High Energy Phys. 09 (2004) 025; D-T. Peng, gr-qc/0501047.
- [8] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
- [9] P. Hořava and E. Witten, Nucl. Phys. **B460**, 506 (1996); P. Hořava and E. Witten, Nucl. Phys. **B475**, 94 (1996).
- [10] A. Lukas, B. A. Ovrut, K. S. Stelle, and D. Waldram, Phys. Rev. D **59**, 086001 (1999).
- [11] T. Kobayashi and T. Tanaka, Phys. Rev. D **69**, 064037 (2004).
- [12] K. Koyama and K. Takahashi, Phys. Rev. D **67**, 103503 (2003); K. Koyama and K. Takahashi, Phys. Rev. D **68**, 103512 (2003).
- [13] S. Mukohyama and A. A. Coley, Phys. Rev. D **69**, 064029 (2004).
- [14] E. Leeper, K. Koyama, and R. Maartens, hep-th/0508145.
- [15] D. S. Salopek and J. R. Bond, Phys. Rev. D **42**, 3936 (1990).
- [16] D. S. Salopek and J. M. Stewart, Classical Quantum Gravity **9**, 1943 (1992).
- [17] J. Parry, D. S. Salopek, and J. M. Stewart, Phys. Rev. D **49**, 2872 (1994).
- [18] H. S. Reall, Phys. Rev. D **59**, 103506 (1999); M. P. Dabrowski, Phys. Lett. B **474**, 52 (2000).
- [19] J. E. Lidsey, Classical Quantum Gravity **17**, L39 (2000); J. E. Lidsey, Phys. Rev. D **62**, 083515 (2000); A. Feinstein, K. E. Kunze, and M. A. Vazquez-Mozo, Phys. Rev. D **64**, 084015 (2001); D. Brecher and M. J. Perry, Nucl. Phys. **B566**, 151 (2000); N. Alonso-Alberca, B. Janssen, and P. J. Silva, Classical Quantum Gravity **17**, L163 (2000).
- [20] C. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).
- [21] A. Peres, Nuovo Cimento **26**, 53 (1962).
- [22] J. E. Lidsey and I. Waga, Phys. Rev. D **51**, 444 (1995).
- [23] S. Kanno and J. Soda, Phys. Rev. D **66**, 043526 (2002); T. Shiromizu and K. Koyama, Phys. Rev. D **67**, 084022 (2003).
- [24] S. Kobayashi and K. Koyama, J. High Energy Phys. 12 (2002) 056.
- [25] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); S. Gubser, I. Klebanov, and A. Polyakov, Phys. Lett. B **428**, 105 (1998).
- [26] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
- [27] V. Balasubramanian and P. Kraus, Phys. Rev. Lett. **83**, 3605 (1999); M. Porrati and A. Starinets, Phys. Lett. B **454**, 77 (1999); D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, Phys. Lett. B **463**, 1 (1999); K. Skenderis and P. K. Townsend, Phys. Lett. B **468**, 46 (1999); O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000).
- [28] S. de Haro, K. Skenderis, and S. Solodukhin, Commun. Math. Phys. **217**, 595 (2001); J. Kalkkinen, D. Martelli, and W. Mück, J. High Energy Phys. 04 (2001) 036; M. Bianchi, D. Freedman, and K. Skenderis, J. High Energy Phys. 08 (2001) 041; M. Bianchi, D. Freedman, and K. Skenderis, Nucl. Phys. **B631**, 159 (2002); K. Skenderis, Classical Quantum Gravity **19**, 5849 (2002); M. Fukuma, S. Matsuura, and T. Sakai, Prog. Theor. Phys. **109**, 489 (2003); D. Martelli and W. Mück, Nucl. Phys. **B654**, 248 (2003); V. Jejjala, R. G. Leigh, and D. Minic, J. Cosmol. Astropart. Phys. 06 (2003) 002.
- [29] J. de Boer, E. Verlinde, and H. Verlinde, J. High Energy Phys. 08 (2000) 003; E. Verlinde and H. Verlinde, J. High Energy Phys. 05 (2000) 034; J. de Boer, Fortschr. Phys. **49**, 339 (2001).
- [30] S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D **62**, 064006 (2000).
- [31] E. Kiritsis, hep-th/0504219.
- [32] S. Nojiri and S. D. Odintsov, Phys. Lett. B **444**, 92 (1998); S. Nojiri, S. D. Odintsov, and S. Ogushi, Gravitation Cosmol. **6**, 271 (2000); M. Fukuma, S. Matsuura, and T. Sakai, Prog. Theor. Phys. **104**, 1089 (2000); K. Koyama and J. Soda, J. High Energy Phys. 05 (2001) 027.
- [33] S. Nojiri and S. D. Odintsov, Int. J. Mod. Phys. A **16**, 3273 (2001); S. Nojiri, S. D. Odintsov, and S. Ogushi, Phys. Rev. D **62**, 124002 (2000); S. Nojiri and S. D. Odintsov, Phys. Lett. B **519**, 145 (2001); S. Nojiri and S. D. Odintsov, Phys. Lett. B **531**, 143 (2002); S. Nojiri and S. D. Odintsov, Int. J. Mod. Phys. A **18**, 2001 (2003); M. Cvetič, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **69**, 023513 (2004).
- [34] H. Verlinde, Nucl. Phys. **B580**, 264 (2000); S. S. Gubser, Phys. Rev. D **63**, 084017 (2001); S. W. Hawking, T. Hertog, and H. S. Reall, Phys. Rev. D **62**, 043501 (2000); S. de Haro, K. Skenderis, and S. N. Solodukhin, Classical Quantum Gravity **18**, 3171 (2001); S. B. Giddings, E. Katz, and L. Randall, J. High Energy Phys. 03 (2000) 023; A. Hebecker and J. March-Russell, Nucl. Phys. **B608**, 375 (2001); T. Shiromizu and D. Ida, Phys. Rev. D **64**, 044015 (2001); T. Shiromizu, T. Torii, and D. Ida, J. High Energy Phys. 03 (2002) 007; B. Chen and F. Lin, Phys. Rev. D **65**, 044007 (2002); T. Kubota, T. Ueno, and N. Yokoi, Phys. Lett. B **579**, 200 (2004).
- [35] H. J. Boonstra, K. Skenderis, and P. K. Townsend, J. High Energy Phys. 01 (1999) 003; K. Behrndt, E. Bergshoeff, R. Halbersma, and J. van der Schaar, Classical Quantum Gravity **16**, 3517 (1999).